A TOPOLOGICAL STUDY OF INDUCED REPRESENTATION*

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INTRODUCTION

In 1939 E. Wigner[1,2] proposed the induced representation technique when he was obtaining the unitary representation of the Poincaré group. This technique is very useful for elementary particle physics. All elementary particles can be corresponded to the induced representations of the Poincaré group. We have not found any exception.

The Poincaré group P is composed of two parts:

$$P \;=\; T \otimes L. \tag{1.1}$$

One part T is the parallel translation in space-time and another part L is the Lorentz transformation. The symbol $\otimes$ means the semi-direct product. We take $\hat{k}_\mu$ to be the generators of the translation, that is, momentum and energy. The

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summation of their square

$$\hat{k}_0^2 - \sum_{i=1}^{3} \hat{k}_i^2 = \hat{\kappa}^2 \tag{1.2}$$

is invariant under the translation and the Lorentz transformation. $\hat{\kappa}^2$ is the Casimir operator of the Poincaré group and so the representations are classified by the eigenvalues of $\hat{\kappa}$ whose physical meaning is the mass of the particle. These classes are as follows:

$$[M_\pm], \quad [0_\pm], \quad [L], \quad [T]$$

where $\pm$ is $\mathrm{sgn}(k_0)$ and $+$ and $-$ indicate the particle and the anti-particle, respectively. The first classes correspond to the usal massive particle and the second ones are the massless particle. We should notice that these classes do not include the zero energy mode, which belongs to the third class $[L]$. The last case is the tachyon.

A vector space is introduced to provide the representation in the concrete. We take the momentum diagonal state $\mid k, \xi >$ as a vector for the representation. Here, ξ means the spin degree. The inner product $< k, \xi \mid k', \xi' >$ must be invariant under the Lorentz transformation.

For the massive case, the Lorentz invariant inner product is given by

$$< k, \xi \mid k', \xi' >= \delta_{\xi\xi'} \omega_k \delta^3(\vec{k} - \vec{k}') \tag{1.3}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. The state vectors are well-defined everywhere on the momentum space, which is equivalent to the 3-dimensional Euclidean space. This space is topologically trivial.

For the massless case, the Lorentz invariant inner product is

$$< k, \xi \mid k', \xi' >= \delta_{\xi\xi'} k_0 \delta^3(\vec{k} - \vec{k}') \tag{1.4}$$

where $k_0 = \mid \vec{k} \mid$. On the occasion of defining the states, one point is subtracted

from the momentum space. Therefore, the state vectors are on the $R^3 - \{0\}$ space, which is homotopically equivalent to S^2. It is a topologically non-trivial space.

There exist many problems in the quantum field theory of the massless particle[3], for example, gauge anomaly[4]. Most problems are related with the zero energy mode, which is subtracted from the region of definition of the state vectors. In order to settle these problems, we must study the induced representaion technique on the topologically non-trivial space and its application to quantum theories.

In 1968, furthermore, E. Mackey[5,6] generalized Wigner's technique to other groups and he study quantm mechanics on a homogeneous space (G/H) by using the technique. In his research he discoursed that there exist many inequivalent quantizations for quntum mechanics on a topologically non-trivial configulation space and that they can be classified according to the representaions of H. This situation does not appear in the usual approach[7]. We introduce a position operator on the homogeneous space and the diagonal state vector of this operator is taken to be one for the representation. But, such vectors is not always defined over the homogeneous space, since it is not guarenteed that vecor fields do not vanish anywhere. We need to study quantum mechanics of the case that the state vector can not be defined over all.

In this note, we will study the problem how Wiger's argument about the induced representaion technique is changed for the topologically non-trivial case. Our talk was in the following order; § 2. Wigner's argument and toplogy, § 3. Euclidean group and gauge structrue and § 4. Quantum mechanics on a sphere.

WIGNER'S ARGUMENT AND TOPOLOGY

Let us review Wigner's argument[1,2] about the induced representation technique. The group, with which we are concerned, is

$$M \ = \ V \otimes G \tag{2.1}$$

where V and G are an abelian group and a transformation group, respectively. The group law on $M = V \otimes G$ is

$$(a_2, \Lambda_2) \cdot (a_1, \Lambda_1) = (a_2 + \Lambda_2 * a_1, \Lambda_2 \Lambda_1). \tag{2.2}$$

The action of G on a is remained on a homogeous space $(G/H \equiv X)$ such as

$$\Lambda * a \in X. \tag{2.3}$$

In order to obtain the representation of G on the space X, we take the following state of the position operator $\hat{x}$ on X,

$$\hat{x} \mid x, \xi >= x \mid x, \xi > . \tag{2.4}$$

These state vectors should be single-valued. If they are degenerate, a new degree ξ is introduced to remove their degeneracy. We consider the representaion matrix $G(\Lambda)$ on the state vector $\mid x, \xi >$. The representaion matrix $G(\Lambda)$ is divided into two parts as

$$G(\Lambda) \;=\; Q(\Lambda, \hat{x}) P(\Lambda) \tag{2.5}$$

where $P(\Lambda)$ and $Q(\Lambda, \hat{x})$ are diagonal for ξ and $\hat{x}$, respectively. They act on the state vector in the following manner,

$$P(\Lambda) \mid x, \xi > = \mid \Lambda * x, \xi >, \quad Q(\Lambda, \hat{x}) \mid x, \xi' > = \mid x, \xi > Q^{\xi\xi'}(\Lambda, x). \tag{2.6}$$

Their product rules

$$P(\Lambda') P(\Lambda) = P(\Lambda' \Lambda), \qquad Q(\Lambda', \hat{x}) Q(\Lambda, \Lambda'^{-1} \hat{x}) \;=\; Q(\Lambda' \Lambda, \hat{x}) \tag{2.7}$$

are derived from $G(\Lambda') G(\Lambda) = G(\Lambda' \Lambda)$, eq.(2.5) and eq.(2.6). $P(\Lambda)$ satisfies the properties of the representation matrix but $Q(\Lambda, \hat{x})$ does not do so.

If $\Lambda'^{-1}\hat{x} = \hat{x}$, $Q(\Lambda', \hat{x})Q(\Lambda, \hat{x}) = Q(\Lambda'\Lambda, \hat{x})$ and then $Q(\Lambda', \hat{x})$ is the representaion matrix. Then, Wigner define a little group by

$$\lambda l \; = \; l, \tag{2.8}$$

where l is some fixed point on X. It is obvious that the little is the subgroup H. Next, he show that any element of G can be reduced to the representation of the little group and the translation on X by taking a suitable unitary transformation

$$G(\Lambda) \; \simeq \; U(\hat{x})G(\Lambda)U^{-1}(\hat{x}) \; = \; U(\hat{x})Q(\Lambda, \hat{x})U^{-1}(\Lambda^{-1}\hat{x})P(\Lambda), \tag{2.9}$$

where $\simeq$ means the unitary equivalence. This unitary matrix $U(\hat{x})$ is determined through the following procedure. The boost transformation is introduced such as

$$\alpha_x l \; = \; \hat{x} \quad (\alpha_x \in G). \tag{2.10}$$

The element λ_x, given by $\alpha_x^{-1}\Lambda\alpha_{\Lambda^{-1}x}$, satisfies $\lambda_x l \; = \; l$ and then any element Λ is expressed by

$$\Lambda \; = \; \alpha_x \lambda_x \alpha_{\Lambda^{-1}x}^{-1}. \tag{2.11}$$

Substituting this form into $Q(\Lambda, \hat{x})$, we have

$$Q(\Lambda, \hat{x}) \; = \; Q(\alpha_x \lambda_x \alpha_{\Lambda^{-1}x}^{-1}, \hat{x}) \; = \; Q(\alpha_x, \hat{x})Q(\lambda_x, l)Q(\alpha_{\Lambda^{-1}x}^{-1}, l). \tag{2.12}$$

Let the unitary matrix $U(\hat{x})$ in eq.(2.9) be $Q(\alpha_x, \hat{x})$. It is easy to see that

$$G(\Lambda) \; \simeq \; Q(\lambda_x, l)P(\Lambda). \tag{2.13}$$

Here, $Q(\lambda_x, l)$ is just a representation matrix of the subgroup H and it is called Wigner rotation. We arrive at the unitary equivalent representation which is divided into the translation and some representaion of the subgroup H.

The boost operator α_x moves a vector at l to one at $\hat{x}$, and then a vector is assigned smoothly to each point of X by α_x. It is called a vector field over X. However, if a d-dimensional manifold M is topologically non-trivial, it is not necessarily possible to define d vector fields which are linearly independent everywhere. When the d-dimensional homogeous space X does not admit d linearly independent vector fields over X, the boost operator α_x is not well defined over X and then we can not determine the unitary matrix $U(\hat{x})$. Moreover, the invariant inner product of the $\hat{x}$-diagonal states includes d δ-functions, but these δ-functions are not defined at a point where d vector fields are not linearly independent. Therefore the state vecors are not defined over all.

Wigner's technique, reviewed in this section, must be amended for the manifold which is not parallelisable. In the next section, we take S^d as a homogeous space and show how Wigner's argument is modified.

EUCLIDEAN GROUP AND GAUGE STRUCTURE

We take the Euclidean group which is composed of the translation and the rotation group $SO(d+1)$ in the d+1 dimensional Euclidean space. When the induced representation is obtained, we do not use the momentum operator but the position operator $\hat{x}$ which is a vector in the d+1 dimensinal Euclidean space. The summation of the square of its components $\hat{x}_i$; $r^2 = \sum_i^{d+1} \hat{x}_i^2$ is invariant under the rotation. r is fixed to be 1. The boost transformation is defined by

$$\alpha_x l = \hat{x}, \qquad \alpha_x \in SO(d+1), \tag{3.1}$$

where l indicates some fixed point on S^d.

This operator also translates the vector fields at l to ones at $\hat{x}$. However, it is well known that the sphere S^d, except for $d = 1, 3$ and 7, does not admit d vector fields which are linearly independent everywhere on S^d. Therefore, the

boost operators are not well defined over all. Moreover, the state vectors, which are the eigenstates of the opsition operator, are not well defined over S^d, since the definition of their inner product includes the δ-functions which can not defined at the point where a vector field vanishes.

Wigner's argument should be madified as follows. Firstly, we take two charts on S^d, which are named N and S, respectively. We introduce two fixed points (l^N and l^S), the boost operators (α_x^N and α_x^S) and the state vectors. The induced representation technique, shown in §2, is applied to the group $SO(d+1)$ on each chrat. The unitary matrix $U(\hat{x})$ in eq(2.9) is given by $Q(\alpha_x^N, \hat{x})$ and $Q(\alpha_x^S, \hat{x})$ on each chart.

Secondly, these representations on each chart must be unitary-equivalently connected at the same point in the overlap region such that

$$Q(\alpha_x^N, \hat{x}) G(\Lambda) Q(\alpha_x^N, \hat{x})^{-1} \simeq Q(\alpha_x^S, \hat{x}) G(\Lambda) Q(\alpha_x^S, \hat{x})^{-1}. \tag{3.2}$$

From the uniraty equivalence $Q(\alpha_x^N, \hat{x})$ and $Q(\alpha_x^S, \hat{x})$ are required to satisfy the condition that $R(\hat{x})$, defined by $Q(\alpha_x^S, \hat{x})^{-1} Q(\alpha_x^N, \hat{x})$, must be a single-valued unitary matrix which is assigned smoothly to each point of the overlap region. $R(\hat{x})$ is rewritten by $Q(\alpha_x^{S-1} \alpha_x^N, l^S)$ and it is obvious that it is not a representation matrix. Then it is difficult to confirm directly that the above condition is satisfied by $R(\hat{x})$.

We comment on the gauge structue in the induced representation technique[8]. The relation of the unitary equivalence at some point is

$$R(\hat{x}) G^N(\Lambda, \hat{x}) R(\hat{x})^{-1} \;=\; G^S(\Lambda, \hat{x}), \tag{3.3}$$

where $G^i(\Lambda, \hat{x})$ $(i = S, N)$ is the representation matrix divided into the translation and the representation of $SO(d)$. Let us consider the infinitesimal rotation ($\Lambda = 1 + \omega$) and the displacement δx on S^d which is given by $x + \delta x = (1 + \omega) x$. From

eq.(3.3), the unitary equivalence of the infinitesimal rotation is expressed by

$$\lim_{\delta x_\mu \to 0} \frac{R(\hat{x})G^N(1+\omega,\hat{x})R(\hat{x})^{-1}-1}{\delta x_\mu} = \lim_{\delta x_\mu \to 0} \frac{G^S(1+\omega,\hat{x})-1}{\delta x_\mu} \qquad (3.4).$$

Noting that

$$\frac{1}{\delta x_\mu} = \frac{1}{\omega_{\mu\nu} x_\nu} = \frac{x_\nu}{x^2}\frac{1}{\omega_{\mu\nu}}, \qquad (3.5)$$

we have

$$R(\hat{x})\partial_\mu R(\hat{x})^{-1} + R(\hat{x})A_\mu^N(\hat{x})R(\hat{x})^{-1} = A_\mu^S(\hat{x}), \qquad (3.6)$$

where

$$R(\hat{x})\partial_\mu R(\hat{x})^{-1} \equiv \frac{x_\nu}{x^2} \lim_{\omega_{\mu\nu}\to 0} \frac{R(\hat{x})R(\hat{x}+\omega\hat{x})^{-1}-1}{\omega_{\mu\nu}}, \qquad (3.7a)$$

$$A_\mu^i(\hat{x}) \equiv \frac{x_\nu}{x^2} \lim_{\omega_{\mu\nu}\to 0} \frac{Q^i(1+\omega,\hat{x})-1}{\omega_{\mu\nu}} \qquad (i = N, S). \qquad (3.7b)$$

∂_μ in eq.(3.7a) means the differential on the sphere. $A_\mu^i(\hat{x})$ can be regarded as gauge potentials on each chart, since eq.(3.6) expresses the gauge transformation.

Now, $R(\hat{x})$ is classified by the homotopy group $\pi_{d-1}(U(n))$, since $R(\hat{x})$ is a map $R : S^{d-1} \to U(n)$ where $U(n)$ is a unitary group. Then the winding (wrapping) number can be defined and if $R(\hat{x})$ satisfy the above condition, the number must be a integer. Let us examine this point for S^2 and S^4 whose homotopy groups are $\pi_1(U(1)) = Z$ and $\pi_3(U(n)) = Z$, respectively. The results for the general cases are published elsewhere[9]. The winding (wrapping) numbers are given by

$$k = \frac{1}{2\pi}\int_{S^1} R^{-1}dR, \qquad (3.8a)$$

$$k = \frac{1}{24\pi^2}\int_{S^3} tr\big(R^{-1}dR \wedge R^{-1}dR \wedge R^{-1}dR\big). \qquad (3.8b)$$

Fortunately, we can calculate these numbers without the explicit form of $R(\hat{x})$ such

that

$$1 + R(\hat{x})^{-1} dR(\hat{x}) = Q(\alpha_x^{S-1}\alpha_x^N, l^S)^{-1} Q(\alpha_{x+dx}^{S-1}\alpha_{x+dx}^N, l^S)$$

$$= Q(\alpha_x^{N-1}\alpha_x^S\alpha_{x+dx}^{S-1}\alpha_{x+dx}^N, \alpha_x^{N-1}\alpha_x^S l^S). \qquad (3.9)$$

Defining χ by $\alpha_x^S\alpha_{x+dx}^{S-1}$, we have

$$1 + R(\hat{x})^{-1} dR(\hat{x}) = Q(\alpha_x^{-1}\chi\alpha_{\chi^{-1}x}, l^N). \qquad (3.10)$$

Then, we can show that $1 + R(\hat{x})^{-1} dR(\hat{x})$ is just the Wigner rotation.

We use the stereographic projection:

$$(S) \quad (\hat{x}_i, \hat{x}_{d+1}) = \frac{1}{1+\hat{y}^2}(2\hat{y}_i, \hat{y}^2 - 1), \qquad (3.11a)$$

$$(N) \quad (\hat{x}_i, \hat{x}_{d+1}) = \frac{1}{1+\hat{z}^2}(2\hat{z}_i, 1 - \hat{z}^2), \qquad (3.11b)$$

where $\hat{y} = \frac{\hat{z}}{\hat{z}^2}$ in the overlap region, and the winding (wrapping) number is written dwon explicitly through the following calculation. Firstly, taking the spinor representation of the group $SO(d+1)$, we express the boost trsnsformation as

$$\alpha_x^S = \frac{1}{\sqrt{1+\hat{y}^2}}(1 + \gamma_{d+1}\sum_{i=1}^{d}\gamma_i\hat{y}_i), \qquad (3.12a)$$

$$\alpha_x^N = \frac{1}{\sqrt{1+\hat{z}^2}}(1 - \gamma_{d+1}\sum_{i=1}^{d}\gamma_i\hat{z}_i), \qquad (3.12b)$$

where γ_j is the Hermitian matrix satisfying the Clifford algebra of order $d+1$. χ

is given by

$$\chi = 1 + i \sum_{i=1}^{d+1} \frac{d\hat{y}_i}{1+\hat{y}^2} \sigma_{id+1}, \tag{3.13}$$

where $\sigma_{ij} = \frac{1}{2i}[\gamma_i, \gamma_j]$. Then, after somewhat lengthy calculation we get

$$\alpha_x^{N-1} \chi \alpha_{\chi^{-1}x} = 1 - i \sum_{j,i=1}^{d} \frac{\hat{z}_i d\hat{z}_j}{\hat{z}^2+1} \sigma_{ji}. \tag{3.14}$$

Next, replacing $\frac{\sigma_{ij}}{2}$ with the generator (S_{ij}) of some representation of the group $SO(d)$, we arrive at

$$Q(\alpha_x^{N-1} \alpha_x^S \alpha_{x+dx}^{S-1} \alpha_{x+dx}^N, l^N) = 1 - i \sum_{j,i=1}^{d} \frac{\hat{z}_i d\hat{z}_j}{\hat{z}^2+1} 2S_{ji}. \tag{3.15}$$

Taking account of the metric, we express the winding (wrapping) numbers on the unit sphere such as

$$k = \frac{1}{\pi} \int_{S^1} \hat{z}_i d\hat{z}_j S_{ij}, \tag{3.16a}$$

$$k = \frac{1}{3\pi^2} \int_{S^3} tr\Big(\hat{z}_{i_1} d\hat{z}_{j_1} S_{i_1 j_1} \wedge \hat{z}_{i_2} d\hat{z}_{j_2} S_{i_2 j_2} \wedge \hat{z}_{i_3} d\hat{z}_{j_3} S_{i_3 j_3} \Big). \tag{3.16b}$$

By the way, it is noteworthy that the winding (wrapping) can be written in terms of the gauge fields defined in eq.(3.7b) such that

$$k = -\frac{1}{4\pi} \int_{S^2} F, \tag{3.17a}$$

$$k = -\frac{1}{8\pi^2} \int_{S^4} tr\Big(F \wedge F \Big), \tag{3.17b}$$

where F is the field tensor of A.

QUNATUM MECHANICS ON A SPHERE

We apply the induced representation technique to quantum mechanics on a d-dimensional sphere ($\simeq SO(d+1)/SO(d)$)[6,8] and formulate its path integral for the transition amplitude by using the semi-classical approximation[10].

A state vector $\mid \psi(t) >$ satisfies the Schrödinger equation

$$i\frac{d}{dt} \mid \psi(t) >= \hat{H} \mid \psi(t) > \tag{4.1}$$

whose formal solution is $\mid \psi(t) >= e^{-i\hat{H}t} \mid \psi(0) >$. The quadratic Casimir operator of $SO(d+1)$ is taken as the Hamiltonian $\hat{H}$, because it is invariant under the translation on X and is simple.

We take two charts on X and the following program is executed on each chart. The coordinate operator $\hat{x}$ is diagonalised and the state vectors are spanned by

$$\mid x, \xi >= G(\Lambda) \mid l, \zeta > \tag{4.2}$$

where $G(\Lambda)$ is the representation matrix of $SO(d+1)$, l is some fixed point and $x = \Lambda * l$.

The transition amplitude $\Im(T, 0)$ for a particle to start at x_0 at $t = 0$ and end up at x_f at $t = T$, is given by

$$\Im(T, 0) \ = < x_f, \xi_f \mid e^{-i\hat{H}T} \mid x_0, \xi_0 > . \tag{4.3}$$

We divide it as

$$< x_f, \xi_f \mid \prod_{n=1}^{N} e^{-i\hat{H}n\tau} \mid x_0, \xi_0 >, \tag{4.4}$$

where $\tau = T/N$. Inserting the identity

$$\sum_{\xi_n} \int d\mu(x_n) \mid x_n, \xi_n >< x_n, \xi_n \mid = \ 1 \tag{4.5}$$

where $d\mu(x)$ is the measure satisfying $d\mu(x) = d\mu(\Lambda * x)$, we have

$$\Im(T,0) \; = \; \sum_{\xi_1} \int d\mu(x_1) \cdots \sum_{\xi_{N-1}} \int d\mu(x_{N-1}) < x_f, \xi_f \mid e^{-i\hat{H}\tau} \mid x_{N-1}, \xi_{N-1} >$$

$$\cdots < x_1, \xi_1 \mid e^{-i\hat{H}\tau} \mid x_0, \xi_0 > . \tag{4.6}$$

Each transition amplitude

$$K(x_{n+1}, \xi_{n+1}; x_n, \xi_n) \; = < x_{n+1}, \xi_{n+1} \mid e^{-i\hat{H}\tau} \mid x_n, \xi_n > \tag{4.7}$$

at infinitesimal interval τ is estimated by using the semi-classical approximation. Noting that $\mid x_n, \xi_n >= G(\Lambda_n) \mid l, \zeta >$ where $\Lambda_n * l = x_n$, we find

$$K(x_{n+1}, \xi_{n+1}; x_n, \xi_n) \; = < x_n, \zeta_{n+1} \mid G^{-1}(\omega) e^{-i\hat{H}\tau} \mid x_n, \xi_n > . \tag{4.8}$$

where $\omega \equiv \Lambda_{n+1}\Lambda_n^{-1}$. Since the Hamiltonian is invariant under the action of G, this is rewritten by

$$K(x_{n+1}, \xi_{n+1}; x_n, \xi_n) \; = < x_n, \xi_{n+1} \mid e^{-i\hat{H}\tau} G^{-1}(\omega) \mid x_n, \zeta_n > \tag{4.9}$$

According to the argument in §2, we have

$$K(x_{n+1}, \xi_{n+1}; x_n, \xi_n) \; = < x_n, \xi_{n+1} \mid e^{-i\hat{H}\tau} P(\omega)^{-1} \mid x_n, \zeta_n > Q^{\xi_n \zeta_n}(\omega, x_n)^{-1}. \tag{4.10}$$

All possible representations of the group $SO(d+1)$, which are laveled by S, are put between $< x_n, \zeta_{n+1} \mid$ and $e^{-i\hat{H}\tau} P(\omega)^{-1}$. The amplitude is now given by

$$K(x_{n+1}, \xi_{n+1}; x_n, \xi_n) \; = \; Q^{\xi_n \zeta_n}(\omega, x_n)^{-1} \sum_S e^{-iH(S)\tau} D^S_{\xi_{n+1}\zeta_n}(\omega), \tag{4.11}$$

where

$$D^S_{\xi_{n+1}\zeta_n}(\omega) \; \equiv < x_n, \xi_{n+1} \mid S >< S \mid P(\omega)^{-1} \mid x_n, \zeta_n > . \tag{4.12}$$

Since the Hamiltonian is quadratic Casimir operator of the group $SO(d+1)$, it is easy to calculate its value for the representations. $D^S_{\xi_{n+1}\zeta_n}(\omega)$ can be expressed

by the Gegenbauer polynomial and $Q^{\xi_n\zeta_n}(\omega, x_n)$ can reduced to the representaion matrix of the little group H by applying Wigner's technique. To carry out the summation of S in eq.(4.11) corresponds to integrating of momentum in the usual path integral formalism based on the canonical quantization. It is very hard to carry out the summation without any approximation.

Here, we take quantum mechanics on S^2 as the simplest example and show its results. The investigation on the other cases are in progress. Now, G and H are $SO(3)$ and $SO(2)(\simeq U(1))$, respectively. The Hamiltonian $\hat{H}$ is given by $\frac{1}{2}\vec{L}^2$ where $\vec{L}$ are the generators of $SO(3)$ and their reprentations are specified by the eigenvalues (j,m) of $\vec{L}^2$ and L_z. Then the transition amplitude is

$$K(x_{n+1}; x_n, s) \; = \; Q^s(\omega, x_n)^{-1} \sum_{j=|s|}^{\infty} \frac{2l+1}{4\pi} e^{(-i\frac{\tau j(j+1)}{2})} d^l_{ss}(\omega), \tag{4.13}$$

where s is an integer and $d^l_{ss}(\omega)$ is a well-known representation function. ω is specified by the inner product $x_{n+1}\cdot x_n \; = \; cos\Theta$ and we use the semi-classical approximation; $\tau \ll 1, \; \omega^2 \simeq O(\tau)$. Then after somewhat lengthy calculations we are let to

$$K(x_{n+1}; x_n, s) \; \simeq \; \frac{1}{2\pi i\tau} Q^s(\omega, x_n)^{-1} e^{(\frac{4i}{\tau}(1-cos\frac{\Theta}{2}))} e^{(-i\tau(\frac{s^2}{2}-\frac{1}{4}))}. \tag{4.14}$$

$Q^s(\omega, x_n)$, by using the semi-classical approximationis, is written in terms of the gauge field discussed in § 3 such that

$$Q^n(\omega, x_n) \; \simeq \; 1 + iA, \tag{4.15}$$

where

$$A \; = \; \frac{-sdx_1x_2}{1+x_3} + \frac{sdx_2x_1}{1+x_3}. \tag{4.16}$$

For general cases, if we use the semi-classical approximation, $Q^{\xi_n\zeta_n}(\omega, x_n)$ can be also written in terms of the gauge fields. For example, the instanton gauge field

appears in S^4. This is pointed out from the different approach[11]. However, it is emphasized that the appearance of gauege fields is not exact but based on the semi-classical approximation.

When we take the limit ($N \to \infty$), the transition amplitude (4.6) is given symbolically by

$$\Im(T,0) \;=\; \int D\mu(x(t))\; e^{i(S_{eff}+S_{top})}, \tag{4.17}$$

where

$$S_{eff} \;=\; \int\limits_0^T dt\Big(\frac{1}{2}\sum_{i=1}^{3}(\frac{dx_i}{dt})^2 + \frac{1}{4} - \frac{s^2}{2}\Big), \tag{4.18a}$$

$$S_{top} \;=\; s\int\limits_0^T dt\Big(\frac{-dx_1}{dt}\frac{x_2}{1+x_3} + \frac{dx_2}{dt}\frac{x_1}{1+x_3}\Big). \tag{4.18b}$$

S_{eff} is the classical action of free particle on S^2 except for $(\frac{1}{4} - \frac{s^2}{2})$.

We can also put the similar calculation in practice on another chart. The different result appears only in S_{top} such that

$$S_{top} \;=\; s\int\limits_0^T dt\Big(\frac{dx_1}{dt}\frac{x_2}{1-x_3} - \frac{dx_2}{dt}\frac{x_1}{1-x_3}\Big). \tag{4.19}$$

Quantum mechanics on S^2 is equal to the quantum dynamics in the background field of s magnetic monopoles within the semi-classical approximation.

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